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# On nonlinear wave envelopes of permanent form near a caustic

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In the vicinity of a caustic of a dispersive wave system, where the group velocity is stationary and hence dispersive effects are relatively weak, the nonlinear Schrödinger equation (NLS) breaks down, and the propagation of the envelope of a finite-amplitude wavepacket is governed by a modified nonlinear Schrödinger equation (MNLS). On the basis of the MNLS, a search for wave envelopes of permanent form is made near a caustic. It is shown that possible solitary wave envelopes satisfy a nonlinear eigenvalue problem. Numerical evidence is presented of symmetric, double-hump solitary-wave solutions. Also, a variety of periodic envelopes are computed. These findings are discussed in connection with previous analytical and numerical work.

## 1. Introduction

Various dispersive wave systems have linear dispersion relations with inflection points at certain wavenumbers. Examples include gravity–capillary water waves (Whitham 1974, §12.1), internal ocean waves (Eckart 1961), and deep-water gravity waves at the cusp lines of a ship (Lighthill 1978, §3.10). At an inflection point of a linear dispersion relation, the group velocity attains a local extremum. In the context of linear wave propagation, the physical significance of this extremum is well known: for fully dispersed waves in the far field, the dominant disturbance is found along the corresponding group line which forms a caustic – a boundary where a transition from oscillatory to exponentially decaying wave behaviour takes place (Lighthill 1978, §4.11). The fact that waves are most pronounced near a caustic according to linear theory provides motivation for exploring the potential significance of finite-amplitude effects. In particular, the possibility of wavepackets with envelopes of permanent form, held together through the combined action of dispersion and nonlinearity, is quite attractive from a physical viewpoint; for example, such nonlinear packets could play an important part in the ocean, in the presence of a double thermocline, where caustics are quite common and are associated with the occurrence of Eckart resonances (Eckart 1961). Also, recent field observations (Brown *et al.* 1989) suggest that solitary wavepackets can be found within the wave pattern generated by a ship, and it is of interest to know whether similar phenomena are likely to occur close to the cusp lines.

In general, the propagation of the envelope of a weakly nonlinear wavepacket is governed by the familiar nonlinear Schrödinger equation (NLS) which, under certain conditions, has solitary- and periodic-wave solutions (see, for example, Whitham 1974, §17.7, 17.8). However, the NLS breaks down in the vicinity of a caustic because the dispersion relation is locally flat so that the dispersive term of the NLS vanishes. To achieve a balance between weak dispersive and nonlinear effects, a higher-order dispersive term needs to be included and, thus, a modified nonlinear Schrödinger equation (MNLS) arises.

To be more specific, consider a wave system with linear dispersion relation $\omega = \omega(k)$. Following Whitham (1974, §17.7), expanding $\omega(k)$ in a Taylor series about the carrier wavenumber $k = k_0$ and carrier frequency $\omega = \omega_0$ gives

$$\omega - \omega_0 = \omega_0'(k-k_0) + \tfrac{1}{2}\omega_0''(k-k_0)^2 + \tfrac{1}{6}\omega_0'''(k-k_0)^3 + \dots . \tag{1}$$

Combining (1) with a finite-amplitude dispersion relation of the form

$$\omega_0 = \omega(k_0) + \epsilon^2|A|^2 + O(\epsilon^4),$$

$\epsilon \ll 1$ being a measure of wave steepness, a nonlinear evolution equation for the wave envelope $A(X, T)$ is obtained:

$$A_T + \omega_0' A_X - \tfrac{1}{2}\mathrm{i}\mu\omega_0'' A_{XX} - \tfrac{1}{6}\mu^2\omega_0''' A_{XXX} + \mathrm{i}\frac{\epsilon^2}{\mu}A^2A^* = O(\mu^3, \epsilon^2, \epsilon^4/\mu), \tag{2}$$

where $X = \mu x$, $T = \mu t$ ($\mu \ll 1$) are the 'slow' envelope variables, and * denotes complex conjugate. In general, $\omega_0'' = O(1)$ so that the leading-order dispersive term in (2) is $O(\mu)$ and balance with the nonlinear term is achieved if $\mu = O(\epsilon)$; in such a case the $O(\mu^2)$ term in (2) is of higher order and (2) reduces to the NLS. On the other hand, in the neighbourhood of a caustic, $\omega_0'' = O(\mu)$, both dispersive terms in (2) are $O(\mu^2)$ and balance with the nonlinear term if $\mu = \epsilon^{\frac{2}{3}}$. Then, in a frame of reference moving with the group velocity $\omega_0'$,

$$X' = X - \omega_0' T, \quad T' = \epsilon^{\frac{4}{3}}T,$$

equation (2) reduces to the MNLS:

$$A_T + \mathrm{i}\beta A_{XX} + \gamma A_{XXX} + \mathrm{i}A^2A^* = 0, \tag{3}$$

where primes have been dropped and

$$\beta = -\frac{1}{2}\frac{\omega_0''}{\mu}, \quad \gamma = -\tfrac{1}{6}\omega_0'''.$$

Note that (3) is uniformly valid through a caustic: exactly at a caustic $\omega_0''$ vanishes so that $\beta = 0$, $\gamma = O(1)$, allowing a balance of nonlinear with dispersive effects; while far from a caustic $|\omega_0''| \gg O(\mu)$ so that $|\beta| \gg O(1)$ and the NLS is recovered in this limit.

A formal derivation of the MNLS (3) was carried out by Jang & Benney (1981) in the context of stratified shear flows, and by Akylas (1987) who examined finite-amplitude effects near the cusp lines of the Kelvin ship-wave pattern. Jang & Benney (1981) attempted to find solitary-wave solutions of (3); to simplify their analysis, they used a transformation which, however, *ab initio* excluded the possibility of solitary waves with more than one hump. They presented perturbation expansions suggesting that single-hump solitary waves exist close to a certain limit which, as will be shown later (see Appendix), is entirely equivalent to the NLS limit. Bryant (1984) numerically computed oblique wave groups with envelopes of permanent form on deep water by substituting truncated modal expansions in the full water-wave equations directly, without invoking the narrow-band assumption ($\mu \ll 1$). He presented two families of periodic wave envelopes, but not solitary envelopes, near the caustic which forms as the group-to-wave angle approaches the critical angle corresponding to waves on the cusp lines of the Kelvin ship-wave pattern; as expected, the periodic- and solitary-wave solutions of the NLS become singular near this critical angle (Hui & Hamilton 1979).

In the present paper, a search for wave envelopes of permanent form is made near a caustic, on the basis of the MNLS. It is shown that possible solitary wave envelopes satisfy a certain nonlinear eigenvalue problem; as a result, unlike the 'sech' solitary waves of the NLS, both the envelope profile and the wave speed are determined when the envelope peak amplitude is specified. Using a shooting procedure, a new class of double-hump, symmetric solitary waves is computed, but no single-hump solitary envelopes are found. There is numerical evidence that, as the NLS limit is approached, these double-hump solitary waves of the MNLS tend to a pair of identical solitary waves of the NLS, located far apart so that their interaction becomes negligible. Furthermore, periodic-envelope solutions of the MNLS are calculated. It is found that, in addition to the two types of envelopes presented by Bryant (1984) near the critical group-to-wave angle, other periodic envelopes with more complicated structure are also possible. Finally, in an Appendix, perturbation expansions of possible single-hump solitary waves of the MNLS close to the NLS limit are discussed, and it is shown that these expansions are equivalent to those of Jang & Benney (1981); it is pointed out that straightforward perturbation theory does not reveal the fact that solitary waves of the MNLS are solutions of an eigenvalue problem.

## 2. Solitary wave envelopes

We look for permanent-wave solutions of (3):

$$A = a(\xi)\exp[\mathrm{i}(K\xi - \Omega T)], \tag{4}$$

where

$$\xi = X - VT,$$

and $K$, $\Omega$, $V$ are constant parameters. Then, upon substitution of (4) into (3), it is found that $a(\xi)$ satisfies

$$\bar{\beta}a_{\xi\xi} - \bar{\Omega}a + a^2a^* + \mathrm{i}\bar{V}a_\xi - \mathrm{i}\gamma a_{\xi\xi\xi} = 0, \tag{5}$$

where

$$\bar{\beta} = \beta + 3\gamma K, \quad \bar{\Omega} = \Omega + KV + \beta K^2 + \gamma K^3, \quad \bar{V} = V + 2\beta K + 3\gamma K^2. \tag{6}$$

It is convenient to normalize variables according to

$$a = |\bar{\Omega}|^{\frac{1}{2}}a', \quad \xi = \frac{\gamma^{\frac{1}{3}}}{|\bar{\Omega}|^{\frac{1}{3}}}\xi', \tag{7a}$$

so that, after dropping the primes, (5) becomes

$$\lambda a_{\xi\xi} - sa + a^2a^* + \mathrm{i}va_\xi - \mathrm{i}a_{\xi\xi\xi} = 0, \tag{7b}$$

where

$$\lambda = \frac{\bar{\beta}}{\gamma^{\frac{2}{3}}|\bar{\Omega}|^{\frac{1}{3}}}, \quad v = \frac{\bar{V}}{\gamma^{\frac{1}{3}}|\bar{\Omega}|^{\frac{2}{3}}}, \quad s \equiv \operatorname{sgn}\bar{\Omega}, \tag{7c}$$

and, without loss of generality, $\gamma$ has been taken to be positive.

Now, attention is focused on calculating solitary-wave solutions. For this purpose, we require

$$r(\xi) \to 0 \quad (\xi \to \pm\infty), \tag{8a}$$

where $a = r\exp(\mathrm{i}\theta)$, so that the wave envelope remains localized. Furthermore, we impose the normalization

$$\kappa \to 0 \quad (\xi \to \infty), \qquad \kappa \to \kappa_0 \quad (\xi \to -\infty), \tag{8b}$$

where $\kappa(\xi) = \theta_\xi$ and $\kappa_0$ is a constant. Note that there is no loss of generality in assuming that $\kappa \to 0$ as $\xi \to \infty$ because, as is clear from (6), any other constant value can be absorbed in the carrier wavenumber of the packet.

In view of condition (8*a*), equation (7*b*) can be linearized, to leading order, at the tails of the packet ($\xi \to \pm\infty$). Accordingly, we write

$$a = f + \mathrm{i}g$$

and we look for asymptotic solutions of (7*b*) as $\xi \to \infty$ in the form

$$f \sim \mathrm{Re}\{\hat{f}\mathrm{e}^{-\sigma\xi}\}, \quad g \sim \mathrm{Re}\{\hat{g}\,\mathrm{e}^{-\sigma\xi}\} \quad (\xi \to \infty); \tag{9}$$

substituting (9) into (7*b*), neglecting the nonlinear term, it is found that

$$(\lambda\sigma^2 - s)^2 + \sigma^2(v - \sigma^2)^2 = 0, \tag{10a}$$

and

$$\frac{\hat{f}}{\hat{g}} = \frac{\lambda\sigma^2 - s}{\sigma(v - \sigma^2)}. \tag{10b}$$

As (10*a*) is a cubic in $\sigma^2$, there are two possibilities: there are either three real or one real and a pair of complex conjugate roots. First, suppose that $\sigma^2$ is real. Clearly $\sigma^2$ is not allowed to be negative because then (8*a*) is violated; furthermore $\sigma^2 > 0$ can be a root of (10*a*) only under the conditions $s\lambda > 0, v > 0$ in which case

$$\sigma = \left(\frac{s}{\lambda}\right)^{\frac{1}{2}}, \quad v = \frac{s}{\lambda}. \tag{11a, b}$$

On the other hand, for a pair of complex conjugate roots of (10*a*), $\sigma = p \pm \mathrm{i}q$ with $p > 0$ so that (8*a*) is satisfied, (9) and (10*b*) give

$$f \sim C\,\mathrm{e}^{-p\xi}\cos(q\xi + \phi), \quad g \sim \pm C\,\mathrm{e}^{-p\xi}\sin(q\xi + \phi) \quad (\xi \to \infty), \tag{12}$$

where $C, \phi$ are constants. Therefore,

$$\kappa = \frac{g_\xi f - g f_\xi}{f^2 + g^2} \sim \pm q \quad (\xi \to \infty),$$

and the first condition in (8*b*) is violated unless $q = 0$ in which case $\sigma$ is real. So, finally, we conclude that the only acceptable value of $\sigma$ is the one given by (11*a*) and this also fixes the speed parameter $v$ according to (11*b*). Furthermore, a similar asymptotic analysis at the other tail of the packet ($\xi \to -\infty$), making use of (11*b*), implies that

$$\kappa_0 = 0,$$

so that the most general asymptotic behaviour at the tails is given by

$$f \sim C_1^{\pm}\exp(\mp(s/\lambda)^{\frac{1}{2}}\xi), \quad g \sim C_2^{\pm}\exp(\mp(s/\lambda)^{\frac{1}{2}}\xi) \quad (\xi \to \pm\infty), \tag{13}$$

where $C_1^{\pm}, C_2^{\pm}$ are constants and $s\lambda > 0$.

On the basis of (13), a shooting procedure can be readily devised to search for symmetric solitary wave envelopes with $f$ even and $g$ odd functions of $\xi$: starting at

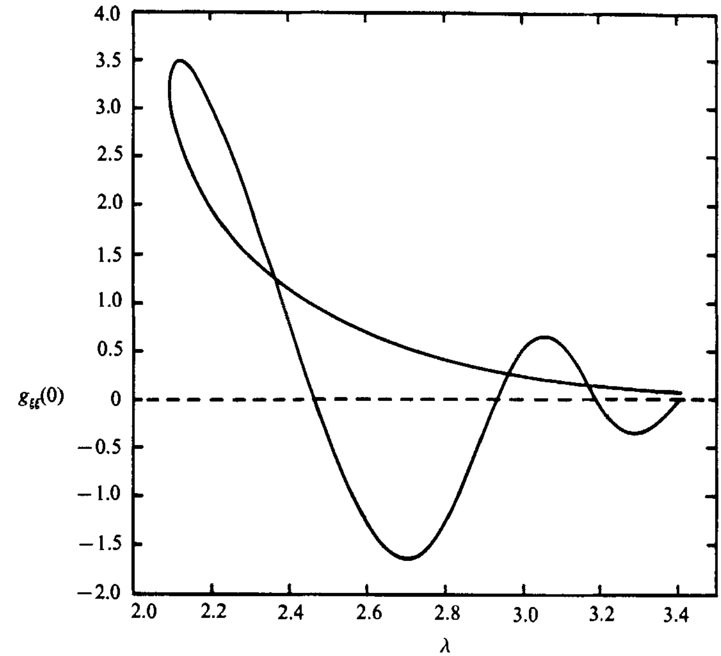


FIGURE 1. Plot of $g_{\xi\xi}(0)$ as a function of the parameter $\lambda$; zeros of $g_{\xi\xi}(0)$ correspond to solitary-wave solutions of the MNLS.

a sufficiently large value of $\xi$, $\xi_\infty$ say, with the asymptotic expressions (13), integrate (7*b*) towards $\xi = 0$ where the continuity conditions

$$f_\xi = 0, \quad g = 0, \quad g_{\xi\xi} = 0 \quad (\xi = 0) \tag{14}$$

are to be imposed. These conditions provide three equations for the three unknown parameters $C_1^+, C_2^+$, and $\lambda$; any possible real solution to these equations corresponds to a solitary wave envelope whose speed parameter $v$ is specified by (11*b*). Thus, it is concluded that solitary waves of the assumed form can exist only for special values of $\lambda$; these are eigenvalues of the nonlinear eigenvalue problem consisting of (7*b*) subject to the boundary conditions (13).

To implement this shooting procedure numerically, it proves convenient to keep $\lambda$ as a free parameter and, through Newton iteration, adjust $C_1^+, C_2^+$ such that the first two of conditions (14) are satisfied; in general, for a given value of $\lambda$, $g_{\xi\xi}(0) \neq 0$, and only eigenvalues $\lambda$ for which $g_{\xi\xi}(0)$ happens to vanish correspond to solitary wave envelopes. A convenient starting point for the Newton iteration is the NLS limit, $\lambda \gg 1$, which is discussed in detail in the Appendix; for $\lambda \gg 1$, as a first guess, $g(\xi)$ can be taken to be zero and $f(\xi)$ to be the NLS solitary wave. After convergence is reached and $g_{\xi\xi}(0)$ is found at a certain value of $\lambda$, a standard continuation procedure, using pseudo-arclength as a continuation parameter (Keller 1977) where necessary, is followed to trace solution branches at other values of $\lambda$.

Figure 1 shows the behaviour of $g_{\xi\xi}(0)$ as a function of $\lambda$. As $\lambda$ is decreased, departing from the NLS limit, $g_{\xi\xi}(0)$ originally increases monotonically so that no solitary envelopes are possible; however, at $\lambda$ around 2.1, a rapid fold takes place beyond which $g_{\xi\xi}(0)$ oscillates between positive and negative values, first becoming zero at $\lambda = 2.465$. The amplitude $|a|$ and wavenumber $\kappa$ of the corresponding solitary wave envelope are shown in figure 2(*a*); this is a double-hump solitary wave with variable

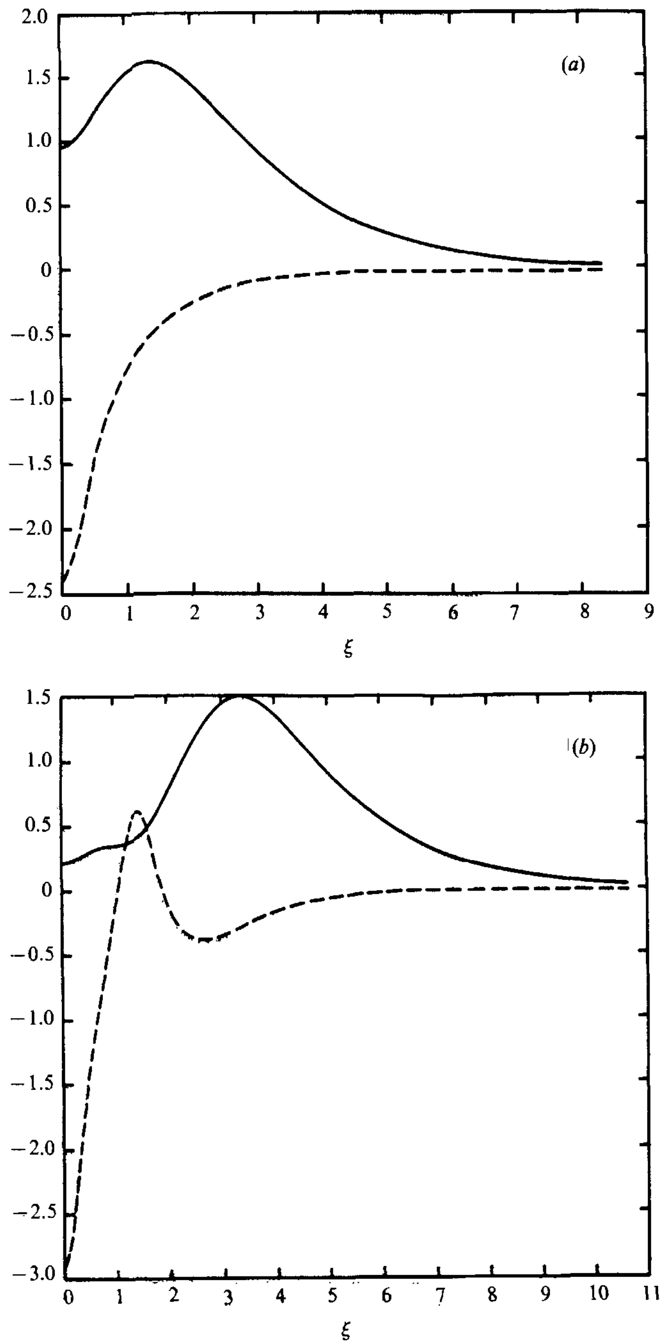


FIGURE 2 (a, b). For caption see facing page.

wavenumber which reaches a minimum at $\xi = 0$. The next two zeros of $g_{\xi\xi}(0)$ occur at $\lambda = 2.9325$, $\lambda = 3.194$ and the corresponding envelopes are plotted in figures 2 (b), 2 (c) respectively; again, we obtain double-hump envelopes, qualitatively similar to the previous one. It is worth noting that as $\lambda$ increases, the amplitude of the trough at $\xi = 0$ decreases rapidly and the separation distance between the two amplitude

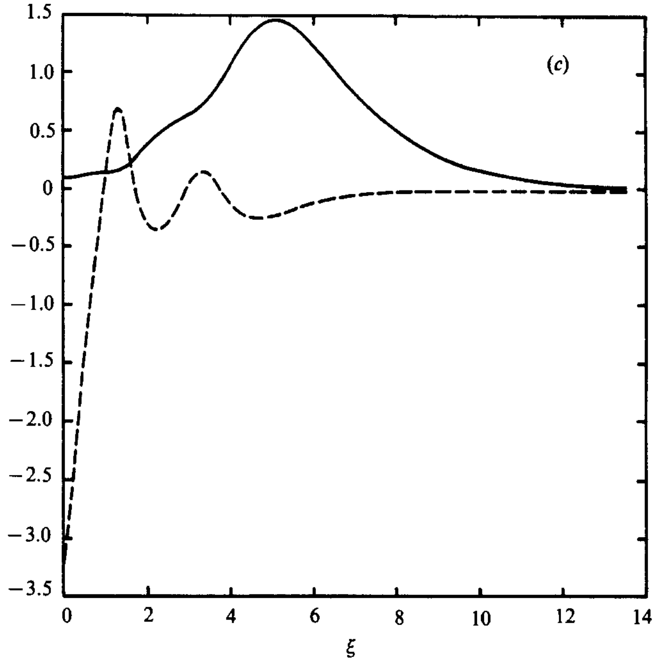


FIGURE 2. Double-hump solitary-wave solutions of the MNLS (only half of the profile is displayed owing to symmetry), ——, envelope amplitude, $|a|$; ----, envelope wavenumber, $\kappa$. (*a*) $\lambda = 2.465$, (*b*) $\lambda = 2.9325$, (*c*) $\lambda = 3.194$.

maxima increases. These trends are consistent with numerical evidence that $g_{\xi\xi}(0)$ has more zeros (perhaps an infinite number of zeros) for larger values of $\lambda$; moreover, the distance between successive zeros decreases, suggesting that, as $\lambda$ increases, the double-hump solitary envelope of the MNLS tends to a pair of well-separated solitary waves of the NLS. This claim is also supported by the observation that, in the same limit, the two maxima of the computed envelopes seem to approach $\sqrt{2}$, the peak amplitude of the NLS solitary wave in the normalized variables chosen here.

The results reported above were obtained by implementing the shooting technique using a fourth-order Runge–Kutta method with step size $\Delta\xi = 0.02$ to integrate (7*b*) numerically, starting at $\xi_\infty = 30$ with the asymptotic solutions (13); further improvement of the resolution did not have any appreciable effect on the numerical results. Also, as an independent check, the calculated envelope profiles were used as initial conditions for the MNLS (3), which was solved numerically through a semi-implicit scheme (Kung 1989); it was verified that these profiles are indeed waves of permanent form propagating with constant speed in accordance with (11*b*).

## 3. Periodic wave envelopes

There are two families of periodic-wave solutions of the NLS; these can be expressed in terms of elliptic functions and are well known (Whitham 1974, §17.7, 17.8; Hui & Hamilton 1979). Envelopes belonging to the first family have two wave groups per period, separated by zeros at which nodes form and the phase jumps by $\pi$ (implying infinite-wavenumber peaks at those points); envelopes in the second family have only one group per period and no zeros, the wave amplitude oscillating periodically between two positive values. For $\beta > 0$, both families are possible and include the NLS solitary wave as a limiting case; for $\beta < 0$, only the first family is possible. As a caustic is approached, $|\beta| \rightarrow 0$, these solutions of the NLS become singular. As noted earlier, Bryant (1984), following a fully numerical approach, was able to compute periodic oblique wave groups in deep water close to the critical group-to-wave angle, where a caustic forms. He presented two families of envelopes, both qualitatively similar to the corresponding solutions of the NLS described above; the only notable difference was that the phase varied continuously in both families, and no envelope zeros were found. Here, the possibility of wavepackets with periodic envelopes close to a caustic is examined on the basis of the MNLS.

We look for symmetric periodic solutions of (7*b*), $a = f + \mathrm{i}g$, with period $2L$; as before, $f$ is taken to be an even and $g$ an odd function of $\xi$ so that the following boundary conditions apply:

$$f_\xi = g = g_{\xi\xi} = 0 \quad (\xi = 0), \tag{15a}$$

$$f_\xi = g = g_{\xi\xi} = 0 \quad (\xi = L). \tag{15b}$$

In addition, we impose the normalization

$$v = 0, \quad f(0) = a_0; \tag{16}$$

here $a_0$ is taken to be known, and the wave profile, as well as the wave period, are to be determined for given values of $\lambda$ and $a_0$.

Two different numerical procedures were used. First, the shooting technique described in §2 can be readily modified to compute periodic waves: starting at $\xi = 0$ with conditions (15*a*), (16) and certain guessed values for $g_\xi(0)$, $f_{\xi\xi}(0)$, equation (7*b*) is integrated numerically towards $\xi = L$, using a fourth-order Runge–Kutta method; through Newton iteration, the appropriate values of $g_\xi(0)$, $f_{\xi\xi}(0)$, and $L$ are determined such that (15*b*) are also met and, thus, the wave profile is found for certain $\lambda$ and $a_0$. Then, solutions at neighbouring values of the parameters $\lambda$, $a_0$ are obtained by continuation. The second numerical method is based on a pseudo-spectral approximation: the computational period $[0, L]$ is discretized by $N+1$ equally spaced points, and $f$, $g$ are interpolated by trigonometric polynomials with the proper periodicity and parity so that conditions (15) are automatically met. Applying the differential equation (7*b*) at the grid points, leads to $2N$ independent nonlinear equations for $2N$ unknowns consisting of $N$ grid values of $f$, $N-1$ grid values of $g$, and $L$. As before, these equations are solved by Newton iteration combined with continuation. In both computational procedures, a first guess for the Newton iteration is provided by the known periodic solutions of the NLS in the limit $|\lambda| \gg 1$. Here, in particular, we choose to start the computation at large positive values of $\lambda$ ($\lambda \gg 1$) with the two families of periodic solutions of the NLS having $s = 1$, $g \equiv 0$, and period equal to $2\pi\lambda^{\frac{1}{2}}$. In this normalization, the maximum envelope amplitudes, according to the NLS, are $a_0 = 1.708462$, $a_0 = 1.39048$ for the first and second family, respectively; these values of $a_0$ are kept fixed as $\lambda$ is varied and the corresponding solution branches are traced numerically.

Figures 3(*a*) and 3(*b*) show typical periodic solutions belonging to the first and second envelope families for $\lambda = 2.8$ and $\lambda = 3.65$, respectively; the corresponding envelope half-periods are $L = 4.828$, $L = 5.360$. Note that both envelope profiles are qualitatively similar to those computed by Bryant (1984): the envelope belonging to the first family is symmetric about $\xi = \frac{1}{2}L$, the two adjacent groups in a period being out of phase, since $f$ is odd and $g$ is even about $\xi = \frac{1}{2}L$, and no nodes are found; envelopes of the second family have only one group per period, as the corresponding

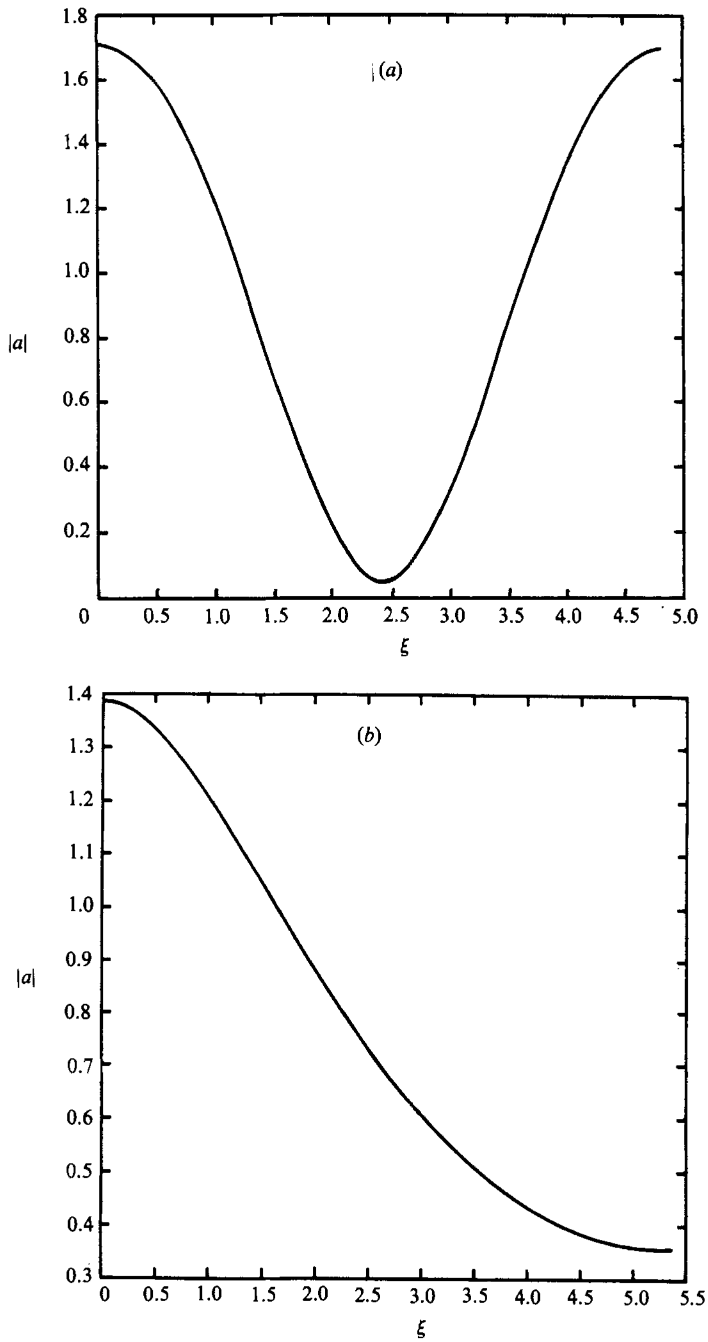


FIGURE 3. Amplitude, $|a|$, of periodic solutions of the MNLS (only half of the profile is displayed owing to symmetry). (*a*) Envelope in the first family with $L = 4.828$ for $\lambda = 2.8$. (*b*) Envelope in the second family with $L = 5.360$ for $\lambda = 3.65$.

solutions of the NLS. However, there is numerical evidence that both envelope families also include other periodic solutions having more wave groups in one period than those predicted by the NLS and computed by Bryant (1984). Here, we make no attempt to discuss all possible solution branches in any detail; we only present two

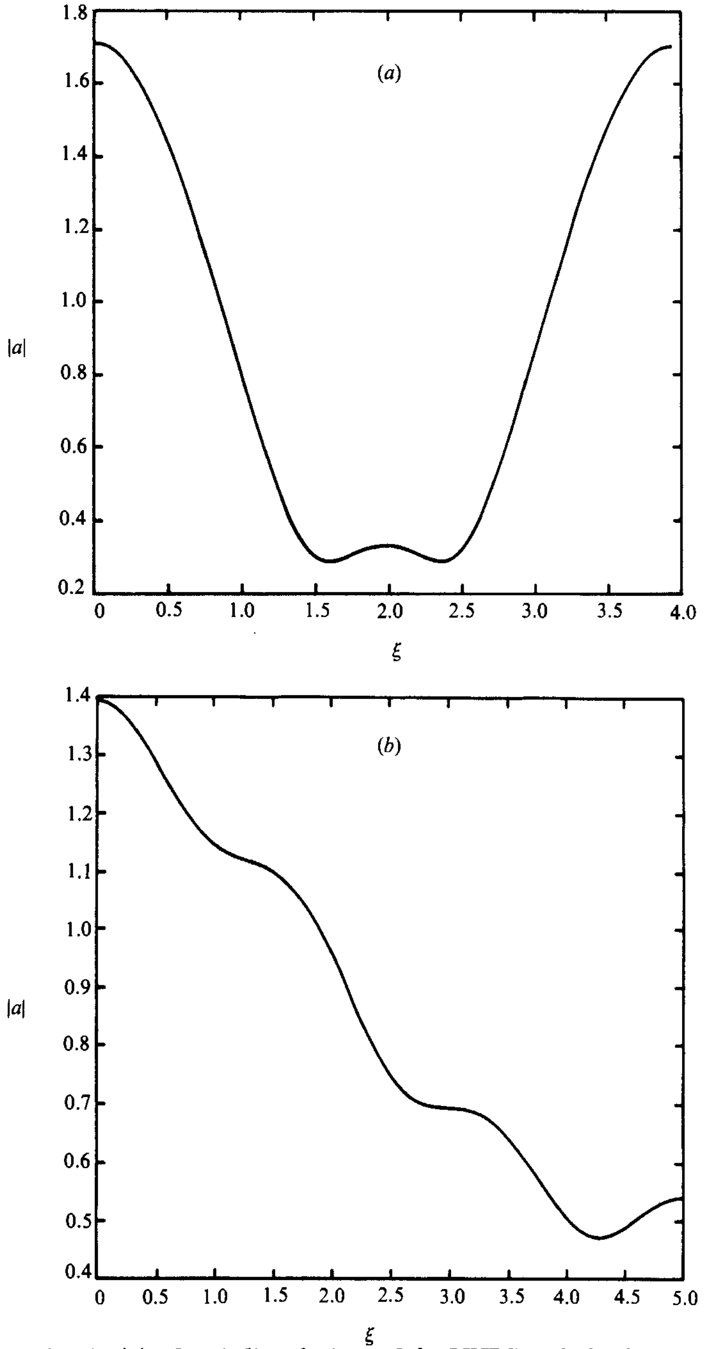


FIGURE 4. Amplitude, $|a|$, of periodic solutions of the MNLS (only half of the profile is displayed owing to symmetry). (*a*) Envelope in the first family with $L = 3.944$ for $\lambda = 2.54$. (*b*) Envelope in the second family with $L = 4.99$ for $\lambda = 3.83$.

examples, one of each family. In fact, the Newton iteration first converged to these solutions when the incremental change of the continuation parameter was not sufficiently small and, as a result, the continuation procedure, accidentally, jumped into a different solution branch. Figure 4(*a*) shows an envelope of the first family for

$\lambda = 2.54$ with $L = 3.944$; as expected, the envelope profile is still symmetric about $\xi = \frac{1}{2}L$ but now there are four wave groups in one period. Similarly, figure 4(*b*) displays an envelope profile of the second family, for $\lambda = 3.83$ with $L = 4.99$, having two wave groups in one period. All numerical solutions were computed using both the shooting method with step size $\Delta\xi = 0.01$ and the pseudo-spectral method with $N = 64$. The two methods gave results in excellent agreement.

The above numerical results suggest that the MNLS has a rich variety of periodic solutions and warrant a detailed study of the corresponding bifurcation diagram.

## 4. Discussion

The analytical and numerical results presented above provide strong evidence that the MNLS admits double-hump solitary-wave solutions, suggesting that nonlinear effects may indeed play an important part near a caustic; under certain conditions, these solitary wavepackets are expected to form the dominant far-field disturbance. As is well known, this is the case for the familiar solitary waves with a 'sech' profile of the NLS. Also, in a different physical context, multi-hump solitary wavepackets of the NLS (in the presence of weak dissipation) have been found to be important in the evolution of modulated cross-waves in a semi-infinite tank (Lichter & Chen 1987; Miles & Becker 1988).

An NLS solitary wave with a 'sech' profile depends on two free parameters (apart from translations in space and time), which may be taken to be the peak amplitude and the speed. On the other hand, as shown in §2, solitary waves of the MNLS are eigensolutions. The scalings (7*a*, *c*) together with (6) and (11*b*) then imply that, for a certain eigenvalue $\lambda$, fixing the peak envelope amplitude (which in turn fixes $\bar{\Omega}$) determines the corresponding solitary envelope completely, including the speed and the wavenumber distribution; that is, MNLS solitary waves form a one-parameter solution family, which suggests that they may not be as easy to obtain from general initial conditions. However, this remains an open question.

Finally, we point out that we were unable to find single-hump solitary waves of the MNLS (see also the discussion in the Appendix). This seems to be consistent with the work of Bryant (1984) who does not report solitary wave groups close to the critical group-to-wave angle.

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## Appendix. Perturbation theory near the NLS limit

As already remarked in §1, the higher-order dispersive term $\gamma A_{XXX}$ of the MNLS becomes negligible away from a caustic ($|\beta| \gg 1$) and (3) reduces to the familiar NLS in this limit. It is well known that for $\beta > 0$, the NLS admits single-hump solitary-wave solutions with a 'sech' profile, and it is of interest to ask whether this is also the case for the MNLS close to the NLS limit. It seems natural to attempt to resolve this question through a perturbation expansion.

Returning to (7*b*), in order to examine the limit $\lambda \gg 1$ it is convenient to rescale

$$\tilde{\xi} = \lambda^{-\frac{1}{2}}\xi, \quad \tilde{v} = \lambda v = O(1), \tag{A 1}$$

so that (7b) (with $s = 1$) becomes

$$a_{\tilde{\xi}\tilde{\xi}} - a + a^2 a^* + \mathrm{i}\alpha(\tilde{v} a_{\tilde{\xi}} - a_{\tilde{\xi}\tilde{\xi}\tilde{\xi}}) = 0, \tag{A 2}$$

where

$$\alpha = \lambda^{-\frac{3}{2}} = \frac{|\bar{\Omega}|^{\frac{1}{2}}\gamma}{\beta^{\frac{3}{2}}}. \tag{A 3}$$

On the assumption that $\alpha$ is small ($\alpha \ll 1$), the perturbation expansions

$$f = f_0 + \alpha f_1 + \alpha^2 f_2 + \dots, \quad g = \alpha g_1 + \alpha^2 g_2 + \dots \tag{A 4}$$

are proposed, where $a = f + \mathrm{i}g$ as before, and

$$f_0 = \surd 2 \operatorname{sech} \tilde{\xi}$$

is the known NLS solitary-wave solution. Substituting (A 4) into (A 2), it is found that, to $O(\alpha)$, $f_1 = 0$ and $g_1$ satisfies the inhomogeneous problem

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tilde{\xi}^2} + 2S^2 - 1\right) g_1 = \frac{1}{\surd 2}[(\tilde{v} - 1) SR + 6S^3 R], \tag{A 5}$$

with the notation $S \equiv \operatorname{sech} \tilde{\xi}$, $R \equiv \tanh \tilde{\xi}$. The right-hand side of (A 5) is an odd function of $\tilde{\xi}$ and it is clearly orthogonal to $S$, which is the well-behaved solution of the corresponding homogeneous problem; therefore (A 5) has a solution which goes to zero as $\tilde{\xi} \to \pm\infty$ and the packet remains localized, in accordance with (8a). However, in order to avoid secular terms which would give rise to non-uniformities in the expansions (A 5) at the tails of the packet, it is necessary, in addition, to insist that the right-hand side of (A 5) is $o(\mathrm{e}^{-|\tilde{\xi}|})$ as $|\tilde{\xi}| \to \infty$. This implies

$$\tilde{v} = 1, \tag{A 6}$$

which, in view of (A 1), is entirely equivalent to (11b). Then, one has

$$g_1 = -\frac{3}{\surd 2} SR.$$

Proceeding to $O(\alpha^2)$, $g_2 = 0$ and, making use of (A 6), $f_2$ satisfies

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tilde{\xi}^2} + 6S^2 - 1\right) f_2 = \frac{9}{\surd 2}(5S^3 - 7S^5). \tag{A 7}$$

This inhomogeneous problem has an acceptable solution because the right-hand side of (A 7) is clearly orthogonal to $RS$, the well-behaved homogeneous solution, and, moreover, it goes to zero faster than $\mathrm{e}^{-|\tilde{\xi}|}$ as $|\tilde{\xi}| \to \infty$ so that no secular terms are generated. In fact,

$$f_2 = \frac{\surd 2}{4}\left[-\frac{39}{2} S + 21 S^3\right].$$

It is clear that the above perturbation procedure can be continued indefinitely without encountering any difficulties: at each successive order, there is an acceptable solution to the corresponding inhomogeneous problem because the parity of the right-hand side is always opposite to the parity of the well-behaved homogeneous solution; also, owing to (A 6), no non-uniformities arise at infinity. This indicates that, for all $\lambda$ close to the NLS limit ($\lambda \gg 1$), the MNLS admits symmetric single-hump solitary-wave solutions, a conclusion that is clearly suspect: according to our earlier argument based on the asymptotic behaviour (13), solitary envelopes are possible only for certain eigenvalues $\lambda$, not for all $\lambda \gg 1$ as suggested by the perturbation theory. In fact, the numerical results presented in §2 seem to suggest that, for single-hump envelopes, $g_{\xi\xi}(0)$ tends to zero from above as $\lambda \rightarrow \infty$, never crossing the $\lambda$-axis for finite $\lambda$ (see figure 1); so it is likely that the MNLS admits no single-hump solitary-wave solutions at all close to the NLS limit. Of course, we emphasize that this is only a suggestion because the numerics are not reliable for arbitrarily large $\lambda$ where $g_{\xi\xi}(0)$ is very small; but, in any event, the perturbation theory is misleading as it gives no evidence of the fact that solitary envelopes are possible only for special values of $\lambda$.

The difficulty of the perturbation theory noted above leads us to conclude that the expansions (A 4) are not convergent in general. Nevertheless, assuming that they are asymptotic, one can possibly interpet these expansions as asymptotic approximations of slightly unsteady wave solutions; this seems plausible in case the unsteady part of the envelope is exponentially small so that it cannot be expanded in powers of $\alpha$. In other problems where similar difficulties arise, it is possible to explicitly calculate the unsteady part, which lies beyond all orders in the asymptotic expansions (Segur & Kruskal 1987). It is desirable to have an improved perturbation theory close to the NLS limit, in order to confirm that solitary-wave solutions of the MNLS exist only for special values of $\lambda$; however, this appears to be a difficult task because it would be necessary to take into account exponentially small terms.

Finally, it is worth clarifying the connection between the perturbation expansions of single-hump solitary waves, proposed by Jang & Benney (1981), and the present work. In looking for solitary-wave solutions of (3), Jang & Benney (1981) set $\gamma = -1$, $\beta = 0$, and allow the envelope wavenumber $\kappa(\xi)$ to have a finite value equal to $K$, say, at the tails of the packet; this choice does not imply loss of generality because, as is clear from (6), it amounts merely to a shift of the carrier wavenumber. Jang & Benney (1981) presented perturbation expansions for the normalized value of $\Omega = 2$ and in the limit $V \rightarrow 3$, $K \rightarrow -1$. Now, according to (6) (with $\beta = 0$, $\gamma = -1$), this limit corresponds to $\bar{\beta} \rightarrow 3$, $\bar{\Omega} \rightarrow 0$, $\bar{V} \rightarrow 0$, so that, in view of (A 3), $\alpha \rightarrow 0$. Therefore, this is the NLS limit and the two perturbation expansions are entirely equivalent; in fact, it is easy to show that the leading-order term of the expansion of Jang & Benney (1981) gives the familiar NLS solitary wave with a 'sech' profile.

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